Ore- and Fan-type heavy subgraphs for Hamiltonicity of 2-connected graphs*

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Abstract

Bedrossian characterized all pairs of forbidden subgraphs for a 2-connected graph to be Hamiltonian. Instead of forbidding some induced subgraphs, we relax the conditions by restricting Ore- and Fan-type degree conditions on these induced subgraphs. Let G be a graph on n vertices and H be an induced subgraph of G. H is called o-heavy if there are two nonadjacent vertices in H with degree sum at least n, and is called f-heavy if for every two vertices $u, v \in V(H)$, $d_H(u, v) = 2$ implies that $\max\{d(u), d(v)\} \geq n/2$. We say that G is H-o-heavy (H-f-heavy) if every induced subgraph of G isomorphic to H is o-heavy (f-heavy). In this paper we characterize all connected graphs R and S other than P_3 such that every 2-connected R- and S-f-heavy (R-o-heavy and S-f-heavy, R-f-heavy and S-free) graph is Hamiltonian. Our results extend several previous theorems on forbidden subgraph conditions and heavy subgraph conditions for Hamiltonicity of 2-connected graphs.

Keywords: Induced subgraphs; o-Heavy subgraphs; f-Heavy subgraphs; Hamiltonicity

AMS Subject Classification (2000): 05C38 05C45

1 Introduction

We use Bondy and Murty [4] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph. For a vertex v and a subgraph H of G, we use $N_H(v)$ to denote the set, and $d_H(v)$ the number, of neighbors of v in H, respectively. We call $d_H(v)$ the *degree* of v in H. For $x, y \in V(G)$, an (x, y)-*path* is a path connecting x and y; the vertex x will be called the *origin* and y the *terminus* of the path. If $x, y \in V(H)$, the *distance*

*Supported by NSFC (No. 11271300).

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between x and y in H, denoted $d_H(x,y)$, is the length of a shortest (x,y)-path in H. If there is no danger of ambiguity, $N_G(v)$, $d_G(v)$ and $d_G(x,y)$ are abbreviated to $N(v)$, $d(v)$ and $d(x,y)$, respectively.

If a subgraph G' of a graph G contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G. For a given graph H, we say that G is *H-free* if G does not contain an induced subgraph isomorphic to H. For a family $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$.

The bipartite graph $K_{1,3}$ is called the *claw*, its (only) vertex of degree 3 is called its *center* and the other vertices are its *end vertices*. In this paper, instead of $K_{1,3}$-free, we use the terminology claw-free.

Many graph theorists drew their attention to find forbidden subgraph conditions for a graph to be Hamiltonian. Bedrossian [1] gave a complete characterization of all pairs of forbidden subgraphs that imply a 2-connected graph is Hamiltonian.

Theorem 1 (Bedrossian [1]). *Let R and S be connected graphs other than P_3 and let G be a 2-connected graph. Then G being $\{R,S\}$-free implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W (see Fig. 1).*

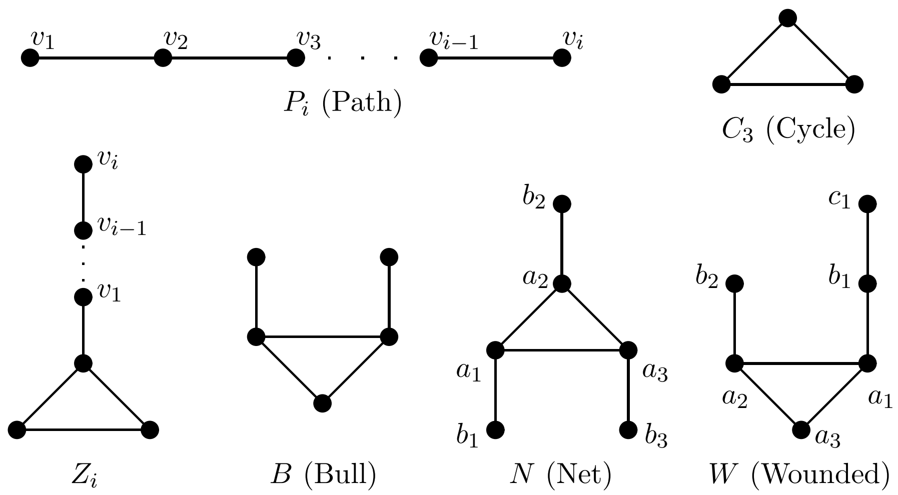

Fig. 1. Graphs P_i, C_3, Z_i, B, N and W.

On the other hand, degree conditions have long been useful tools in the study of Hamilton cycles. Among all, Ore's condition [14] is fundamental.

Theorem 2 (Ore [14]). *Let G be a graph on $n \geq 3$ vertices. If the degree sum of every pair of nonadjacent vertices in G is at least n, then G is Hamiltonian.*

Let G be a graph on n vertices. For a given graph H, we say that G is *H-o-heavy* if for every induced subgraph G' of G isomorphic to H, there exist two nonadjacent vertices

$x, y \in V(G')$ such that $d(x)+d(y) \geq n$. For a family $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$-o-heavy* if G is H-o-heavy for every $H \in \mathcal{H}$. Clearly, if H' is an induced subgraph of H, then every H'-o-heavy graph is always H-o-heavy, and an H-free graph is also H-o-heavy. In this paper, we use the terminology claw-o-heavy instead of $K_{1,3}$-o-heavy.

By relaxing forbidden subgraph conditions to conditions in which the subgraphs are allowed, but where Ore's condition is imposed on these subgraphs if they appear, Li et al. [10] extended Theorem 1 as follows.

Theorem 3 (Li, Ryjáček, Wang and Zhang [10]). *Let R and S be connected graphs other than P_3 and let G be a 2-connected graph. Then G being $\{R, S\}$-o-heavy implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, C_3, Z_1, Z_2, B, N$ or W.*

One may notice that there is only one graph P_6 that appears in Bedrossian's result but miss here. Li et al. [10] also constructed a 2-connected claw-free P_6-o-heavy graph which is not Hamiltonian. With a little effort, they got

Theorem 4 (Li, Ryjáček, Wang and Zhang [10]). *Let S be a connected graph other than P_3 and let G be a 2-connected claw-o-heavy graph. Then G being S-free implies G is Hamiltonian if and only if $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

There is another degree condition due to Fan [8] (so called Fan's condition) with respect to Hamilton cycles.

Theorem 5 (Fan [8]). *Let G be a 2-connected graph on n vertices. If $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v with $d(u, v) = 2$, then G is Hamiltonian.*

Let G be a graph on n vertices. For a given graph H, we say that G is *H-f-heavy* if for every induced subgraph G' of G isomorphic to H, and two vertices $u, v \in V(G')$, $d_{G'}(u, v) = 2$ implies that $max\{d(u), d(v)\} \geq n/2$. In contrast to the case of forbidden subgraphs or o-heavy subgraphs, if H' is an induced subgraph of H, then an H'-f-heavy graph is not always H-f-heavy. For example, Z_2 is always an induced subgraph of W, but a Z_2-f-heavy graph is not necessarily W-f-heavy. For a family $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$-f-heavy* if G is H-f-heavy for every $H \in \mathcal{H}$. Note that an H-free graph is also H-f-heavy. Also, if $H = K_{1,3}$, then we use the terminology claw-f-heavy instead of $K_{1,3}$-f-heavy.

For a given graph $H \in \{P_4, P_5, P_6, C_3, Z_1, Z_2, B, N, W\}$, it is interesting to compare H-o-heavy graphs with H-f-heavy graphs. It is not difficult to see that there exist H-o-heavy graphs which are not H-f-heavy, and H-f-heavy graphs which are not H-o-heavy. Figure 2. shows a graph which is N-f-heavy but not N-o-heavy, W-o-heavy and W-f-heavy.

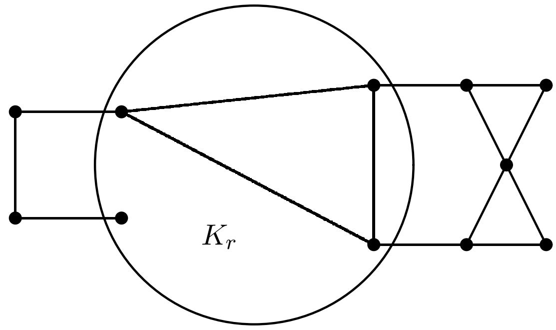

Fig. 2 A graph ($r \geq 7$) which is N-f-heavy but not N-o-heavy, W-o-heavy and W-f-heavy.

Our first aim in this paper is to find corresponding Fan-type heavy subgraph conditions which extend Theorem 1. By Theorem 5, we know that every 2-connected P_3-f-heavy graph is Hamiltonian. It is easy to see that P_3 is the only connected graph S such that every 2-connected S-f-heavy graph is Hamiltonian. Thus we have the following problem.

Problem 1. Which two connected graphs R and S other than P_3 imply that every 2-connected $\{R, S\}$-f-heavy graph is Hamiltonian?

By Theorem 1, we get that (up to symmetry) $R = K_{1,3}$ and S must be one of the graphs $P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.

In fact, there are many previous results [2, 5, 6, 12] which are related to Problem 1, although stated in different terminology and notations.

Theorem 6 (Chen, Wei and Zhang [6]). *Let G be a 2-connected graph. If G is $\{K_{1,3}, P_6\}$-f-heavy, then G is Hamiltonian.*

Theorem 7 (Bedrossian, Chen and Schelp [2]). *Let G be a 2-connected graph. If G is $\{K_{1,3}, Z_1\}$-f-heavy, then G is Hamiltonian.*

Theorem 8 (Li, Wei and Gao [12]). *Let G be a 2-connected graph. If G is $\{K_{1,3}, B\}$-f-heavy, then G is Hamiltonian.*

Theorem 9 (Chen,Wei and Zhang [5]). *Let G be a 2-connected graph. If G is $\{K_{1,3}, N\}$-f-heavy, then G is Hamiltonian.*

In this paper, we prove the following two results.

Theorem 10. *Let G be a 2-connected graph. If G is $\{K_{1,3}, Z_2\}$-f-heavy, then G is Hamiltonian.*

Theorem 11. *Let G be a 2-connected graph. If G is $\{K_{1,3}, W\}$-f-heavy, then G is Hamiltonian.*

On the other hand, a useful remark is shown in the following.

Remark 1. Since C_3 is a clique, it contains no pairs of vertices with distance 2. In this means, we say that every graph is C_3-f-heavy. On the other hand, there indeed exist a 2-connected claw-free graph which is not Hamiltonian (a 3-connected claw-free non-Hamiltonian graph is shown in [13]). Thus, not every 2-connected $\{K_{1,3}, C_3\}$-f-heavy graph is Hamiltonian.

Note that every P_i-f-heavy ($i = 4, 5$) graph is P_6-f-heavy, every Z_1-f-heavy graph is B-f-heavy (N-f-heavy) and every B-f-heavy graph is N-f-heavy. Together with Remark 1 and Theorems 6, 9, 10 and 11, we have

Theorem 12. *Let R and S be connected graphs other than P_3 and let G be a 2-connected graph. Then G being $\{R, S\}$-f-heavy implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.*

Theorem 12 gives a complete answer to Problem 1.

Moreover, we can pose the following two problems naturally.

Problem 2. Which two connected graphs R and S other than P_3 imply that every 2-connected R-o-heavy and S-f-heavy graph is Hamiltonian?

Problem 3. Which two connected graphs R and S other than P_3 imply that every 2-connected R-free and S-f-heavy graph is Hamiltonian?

By Theorem 1, Problem 2 is equivalent to the following two problems.

Problem 2.1. Which connected graphs S other than P_3 imply that every 2-connected claw-o-heavy and S-f-heavy graph is Hamiltonian?

Problem 2.2. Which connected graphs S other than P_3 imply that every 2-connected claw-f-heavy and S-o-heavy graph is Hamiltonian?

For Problem 2.1, by Theorem 1 and Remark 1, we know that S must be one of the graphs $P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.

In this paper, instead of Theorems 10 and 11, we prove the following stronger result.

Theorem 13. *Let G be a 2-connected graph. If G is claw-o-heavy and S-f-heavy, where $S \in \{P_6, Z_2, W, N\}$, then G is Hamiltonian.*

As a corollary of Theorems 1 and 13, we can get the following theorem, which gives a full answer to Problem 2.1.

Theorem 14. *Let G be a 2-connected graph and S be a connected graph other than P_3. If G is claw-o-heavy, then G being S-f-heavy implies G is Hamiltonian if and only if $S = P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.*

For Problem 2.2, we firstly notice that every claw-f-heavy graph is also claw-o-heavy. Secondly, it is known in [10] that there exists a 2-connected claw-free and P_6-o-heavy graph which is not Hamiltonian. Thus the following result, which can be deduced from Theorem 4, is an answer to Problem 2.2.

Corollary 1. Let G be a 2-connected graph and S be a connected graph other than P_3. If G is claw-f-heavy, then G being S-o-heavy implies G is Hamiltonian if and only if $S = P_4, P_5, C_3, Z_1, Z_2, B, N$ or W.

Similar to Problem 2, by Theorem 1, Problem 3 is equivalent to the following two problems.

Problem 3.1. Which connected graphs S other than P_3 imply that every 2-connected claw-free and S-f-heavy graph is Hamiltonian?

Problem 3.2. Which connected graphs S other than P_3 imply that every 2-connected claw-f-heavy and S-free graph is Hamiltonian?

For a given connected graph H, we notice that every H-free graph is also H-f-heavy. Hence by Theorems 4, 12 and Remark 1, we have

Corollary 2. Let G be a 2-connected graph and S be a connected graph other than P_3. If G is claw-free, then G being S-f-heavy implies G is Hamiltonian if and only if $S = P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.

Corollary 3. Let G be a 2-connected graph and S be a connected graph other than P_3. If G is claw-f-heavy, then G being S-free implies G is Hamiltonian if and only if $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.

Corollaries 2 and 3 answer Problems 3.1 and 3.2, respectively.

Obviously, Theorem 13 extends Theorems 6-11, and Theorem 14 extends Theorem 4. Moreover, each of Theorems 12, 14 and Corollaries 1-3 extends Theorem 1.

In the next Section, we give some preliminaries. The proof of Theorem 13 is postponed to Section 3. In the last section, some remarks and one open problem are given.

2 Preliminaries

We begin this section with some additional terminology and notation.

Let G be a graph, H a subgraph of G and X a subset of $V(G)$. We use $G[X]$ to denote the subgraph of G induced by X, and $G-H$ denotes the subgraph induced by $V(G)\setminus V(H)$. If G' is a graph, then by $G[X]\cong G'$, we mean that $G[X]$ is isomorphic to the graph G'.

Throughout this paper, k and l will denote positive integers, and s,t denote the integers which may be non-positive. For $s\leq t$, $[s,t]$ denotes the integer set $\{s,s+1,\ldots,t-1,t\}$ and $[u_s,u_t]$ denotes the set $\{u_s,u_{s+1},\ldots,u_{t-1},u_t\}$. If $[u_s,u_t]$ is a subset of the vertex set of a graph G, we use $G[u_s,u_t]$, instead of $G[[u_s,u_t]]$, to denote the subgraph induced by $[u_s,u_t]$ in G.

Let P be a path and $u,v\in V(P)$. We use $P[u,v]$ to denote the subpath of P from u to v. Let C be a cycle. We denote by $\overrightarrow{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the same subgraph with the reverse orientation. For two vertices $u,v\in V(C)$, $\overrightarrow{C}[u,v]$ denotes the path from u to v on $\overrightarrow{C}$, and $\overleftarrow{C}[v,u]$ is the same path with the reverse direction. For a vertex $x\in\overrightarrow{C}$, we use x^+ to denote the successor of x on $\overrightarrow{C}$, and x^- denotes its predecessor. If $A\subseteq V(C)$, then set $A^+=\{x^+ : x\in A\}$ and $A^-=\{x^- : x\in A\}$.

Let G be a graph on n vertices and v be a vertex of $V(G)$. The vertex v is called *heavy* if its degree is at least $n/2$; otherwise we call it a *light* vertex. A pair of nonadjacent vertices with degree sum at least n is called a *heavy pair* and a triangle such that every vertex in it is heavy is called a *heavy triangle*. A cycle C of G is called *heavy* if it contains all heavy vertices of G; it is called *nonextendable* if there is not a longer cycle in G which contains all the vertices of C.

In this paper, we need some concepts firstly introduced by Li et al. in [10]. To ensure the integrity of our paper, we rewrite them here.

Let G be a graph and $C=x_1x_2,\ldots,x_t$ be a sequence of vertices in $V(G)$, where $t\geq 3$ be an integer. We denote $\widetilde{E}(G)=\{xy : xy\in E(G) \text{ or } d(x)+d(y)\geq n, x,y\in V(G)\}$, and say that C is an *Ore-cycle*, or in short, *o-cycle*, if the vertices in $V(C)$ satisfy $x_ix_{i+1}\in\widetilde{E}(G)$, $i\in[1,t]$, where $x_1=x_{t+1}$.

Let G be a graph and $\{x_1,x_2\}$, $\{y_1,y_2\}$ be two pairs of vertices in $V(G)$ with $x_1\neq x_2$ and $y_1\neq y_2$. We say that D is an *(x_1x_2,y_1y_2)-pair* if D consists of two vertex-disjoint paths P_1 and P_2 such that

(i) the origin of P_i is in $\{x_1,x_2\}$, and

(ii) the terminus of P_i is in $\{y_1,y_2\}$

for $i=1,2$.

Let G be a graph on $n\geq 2$ vertices and $x,y\in V(G)$ be two distinct vertices. Let G' be a graph obtained by adding a (new) vertex z to G with two edges zx and zw, where $w\neq x$ is an arbitrary vertex of G. Let G'' be a graph obtained by adding two (new)

distinct vertices x' and y' to G and three edges xx', yy' and $x'y'$. We call G a *1-extension of* G *from* x *to* z, and G'' a *2-extension of* G *from* (x, y) *to* (x', y').

Let G be a graph and x, y, z be three distinct vertices of $V(G)$. G is called (x, y, z)-*composed* if there exists a sequence of vertices $v_{-k}, \dots, v_0, \dots, v_l$ $(k, l \geq 1)$ and a sequence of graphs $D_1, D_2, \dots, D_r$ $(r \geq 1)$ such that

(1) $x = v_{-k}$, $y = v_0$ and $z = v_l$,

(2) D_1 is a triangle such that $V(D_1) = \{v_{-1}, v_0, v_1\}$,

(3) $V(D_i) = [v_{-x_i}, v_{y_i}]$ for some x_i, y_i, where $1 \leq x_i \leq k$ and $1 \leq y_i \leq l$, and D_{i+1} satisfies one of the following conditions for $i \in [1, r-1]$:

  (i) D_{i+1} is a 1-extension of D_i from v_{-x_i} to v_{-x_i-1} or from v_{y_i} to v_{y_i+1},

  (ii) D_{i+1} is a 2-extension of D_i from (v_{-x_i}, v_{y_i}) to (v_{-x_i-1}, v_{y_i+1}),

(4) $D = D_r$ satisfies $V(D) = V(G)$.

Without loss of generality, we call the sequence of vertices $v_{-k}, \dots, v_0, \dots, v_l$ a *canonical ordering* and the sequence of graphs $D_1, D_2, \dots, D_r$ a *canonical sequence of* D. We call the graph D the *carrier of* G.

Let G be a graph, C a cycle of G and x_1, x, x_2 three distinct vertices on C. Let P be the (x_1, x_2)-path on C such that $x \in V(P) \setminus \{x_1, x_2\}$. The pair of vertices (x_1, x_2) is said to be x-*good on* C, if for some integer $i \in \{1, 2\}$, there exits a vertex $x' \in V(P) \setminus \{x_i\}$ such that

(i) there is an (x, x_{3-i})-path P' such that $V(P') = V(P) \setminus \{x_i\}$,

(ii) there is an $(xx_{3-i}, x'x_i)$-pair D such that $V(D) = V(P)$,

(iii) the degree sum of x_i and x' is at least n.

Next, we list several known results needed in our proof.

Lemma 1 (Li and Zhang [10, 11])**.** *Let* G *be a graph and* C' *be an o-cycle of* G. *Then there is a cycle* C *of* G *such that* $V(C') \subseteq V(C)$.

Lemma 2 (Li and Zhang [11])**.** *Let* G *be a 2-connected* $K_{1,4}$*-o-heavy graph and* C *be a longest cycle of* G. *Then* C *is a heavy cycle of* G.

Lemma 3 (Chvátal and Erdös [7], Bondy [3])**.** *Let* G *be a graph on* n *vertices,* $\overrightarrow{C}$ *a nonextendable cycle in* G, H *a component of* $G - V(C)$, *and* A *the set of neighbours of* H *on* C. *Then*

(i) $A \cap A^- = \emptyset$, $A \cap A^+ = \emptyset$, *and* A^- *and* A^+ *are independent sets,*

(ii) *Each pair of vertices from* A^- *or* A^+ *has degree sum smaller than* n.

Lemma 4 (Li, Ryjáček, Wang and Zhang [10])**.** *Let* G *be a composed graph and let* D *and* $v_{-k}, \dots, v_0, \dots, v_l$ *be a carrier and a canonical ordering of* G. *Then*

(*i*) *D has a Hamilton (v_0, v_{-k})-path,*

(*ii*) *for every $v_s \in V(G) \setminus \{v_{-k}\}$, D has a spanning (v_0v_l, v_sv_{-k})-pair.*

Lemma 5 (Li, Ryjáček, Wang and Zhang [10])**.** *Let G be a graph, and C be a cycle of G with a given orientation. Let P be an (x, y)-path of G which is internally disjoint from C, where $x, y \in V(C)$. If there are vertices $x_1, x_2, y_1, y_2 \in V(C) \setminus \{x, y\}$ such that*

(*i*) *x_2, x, x_1, y_1, y, y_2 appear in the order along $\overrightarrow{C}$ (maybe $x_1 = y_1$ or $x_2 = y_2$);*

(*ii*) *(x_1, x_2) is x-good on C; and*

(*iii*) *(y_1, y_2) is y-good on C,*

then there is a cycle in G which contains all the vertices in $V(C) \cup V(P)$.

3 Proof of Theorem 13

Suppose that G is a non-Hamiltonian graph on n vertices. Let C be a longest cycle of G and c be the length of C. Then $c < n$ and $G - C \neq \emptyset$. Since G is 2-connected, there is a path of length at least 2, internally-disjoint with C, that connects two vertices of C. Let $P = w_0w_1 \dots w_rw_{r+1}$ be such a path with r is as small as possible, where $w_0 = u_0 \in V(C)$ and $w_{r+1} = v_0 \in V(C)$. Assume that the length of $\overrightarrow{C}[u_0, v_0]$ is $r_1 + 1$ and the length of $\overrightarrow{C}[v_0, u_0]$ is $r_2 + 1$. Obviously, $r_1 + r_2 + 2 = c$. We denote the cycle C with a given orientation by $\overrightarrow{C} = u_0u_1 \dots u_{r_1}v_0u_{-r_2} \dots u_{-1}u_0$ or by $\overrightarrow{C} = v_0v_1 \dots v_{r_2}u_0v_{-r_1} \dots v_{-1}v_0$, where $u_l = v_{-r_1-1+l}$ and $u_{-k} = v_{r_2+1-k}$.

Claim 1. Let $x \in [w_1, w_r]$ and $y \in \{u_{-1}, u_1, v_{-1}, v_1\}$. Then $xy \notin \widetilde{E}(G)$.

Proof. Without loss of generality, assume that $y = u_{-1}$. Suppose that $xy \in \widetilde{E}(G)$. Then $C' = P[u_0, x]xy\overleftarrow{C}[y, u_0]$ is an o-cycle containing all the vertices of C and longer than C. By Lemma 1, there is a cycle longer than C, contradicting to the choice of C. □

Claim 2. $u_{-1}u_1 \in \widetilde{E}(G)$, $v_{-1}v_1 \in \widetilde{E}(G)$.

Proof. Assume that $u_{-1}u_1 \notin E(G)$. By Claim 1, we have $w_1u_{-1} \notin E(G)$ and $w_1u_1 \notin E(G)$. Hence $G[\{u_{-1}, u_0, u_1, w_1\}] \cong K_{1,3}$. Note that $w_1u_{-1} \notin \widetilde{E}(G)$ and $w_1u_1 \notin \widetilde{E}(G)$ by Claim 1. Since G is claw-o-heavy, it follows that $d(u_{-1})+d(u_1) \geq n$. Thus, we obtain $u_{-1}u_1 \in \widetilde{E}(G)$.

Similarly, we can prove $v_{-1}v_1 \in \widetilde{E}(G)$. □

Claim 3. $u_0v_{\pm 1} \notin \widetilde{E}(G)$, $v_0u_{\pm 1} \notin \widetilde{E}(G)$.

Proof. Assume that $u_0v_1 \in \widetilde{E}(G)$ or $u_0v_{-1} \in \widetilde{E}(G)$. Let $C' = P\overleftarrow{C}[v_0, u_1]u_1u_{-1}\overleftarrow{C}[u_{-1}, v_1]v_1 u_0$ (if $u_0v_1 \in \widetilde{E}(G)$) or $C' = P\overrightarrow{C}[v_0, u_{-1}]u_{-1}u_1\overrightarrow{C}[u_1, v_{-1}]v_{-1}u_0$ (if $u_0v_1 \notin \widetilde{E}(G)$). By

Claim 2, C' is an o-cycle containing all the vertices in C and longer than C, a contradiction by Lemma 1.

Similarly, we can prove $v_0u_{\pm 1} \notin \widetilde{E}(G)$. $\square$

Let u_{j_1} be the first vertex on $\overrightarrow{C}[u_1, u_{r_1}]$ such that $u_0u_{j_1} \notin E(G)$, v_{j_2} be the first vertex on $\overrightarrow{C}[v_1, v_{r_2}]$ such that $v_0v_{j_2} \notin E(G)$. Obviously, we have $u_0u_1 \in E(G)$ and $v_0v_1 \in E(G)$. By Claim 3, we know that $u_0u_{r_1} \notin E(G)$ and $v_0v_{r_2} \notin E(G)$. Thus, u_{j_1}, v_{j_2} exist, where $2 \leq j_1 \leq r_1$ and $2 \leq j_2 \leq r_2$.

Claim 4. Let $w \in [w_1, w_r], u \in [u_1, u_{j_1}]$ and $v \in [v_1, v_{j_2}]$. Then we have

(i) $wu \notin \widetilde{E}(G), wv \notin \widetilde{E}(G)$,

(ii) $v_0u \notin \widetilde{E}(G), u_0v \notin \widetilde{E}(G)$,

(iii) $uv \notin \widetilde{E}(G)$.

Proof. (i) Assume that $wu \in \widetilde{E}(G)$. If $u = u_1$, then we get a contradiction by Claim 1. If $u = u_2$, then let $C' = P[u_0, w]wu\overrightarrow{C}[u, u_{-1}]u_{-1}u_1u_0$. If $u \in [u_3, u_{j_1}]$, then let $C' = P[u_0, w]wu\overrightarrow{C}[u, u_{-1}]u_{-1}u_1\overrightarrow{C}[u_1, u^-]u^-u_0$. By Claim 2, C' is an o-cycle longer than C and contains all the vertices in C. Therefore, there is a cycle longer than C by Lemma 1, a contradiction.

The second assertion can be proved similarly.

(ii) Assume that $v_0u \in \widetilde{E}(G)$. By Claim 3, we have $v_0u_1 \notin \widetilde{E}(G)$. Hence we have $u \in [u_2, u_{j_1}]$. Then $C' = v_0u\overrightarrow{C}[u, v_{-1}]v_{-1}v_1\overrightarrow{C}[v_1, u_{-1}]u_{-1}u_1\overrightarrow{C}[u_1, u^-]u^-u_0P$ is an o-cycle longer than C and contains all the vertices in C by Claim 2. By Lemma 1, there is a cycle longer than C, contradicting to the choice of C.

Similarly, we can prove that $u_0v \notin \widetilde{E}(G)$.

(iii) Assume that $uv \in \widetilde{E}(G)$. By Claim 2, we have $u_{-1}u_1 \in \widetilde{E}(G)$ and $v_{-1}v_1 \in \widetilde{E}(G)$. Then $C' = Pv_0v^-\overleftarrow{C}[v^-, v_1]v_1v_{-1}\overleftarrow{C}[v_{-1}, u]uv\overrightarrow{C}[u, u_{-1}]u_{-1}u_1\overrightarrow{C}[u_1, u^-]u^-u_0$ (if $u \neq u_1$ and $v \neq v_1$) or $C' = Pv_0v^-\overleftarrow{C}[v^-, v_1]v_1v_{-1}\overleftarrow{C}[v_{-1}, u_1]u_1v\overrightarrow{C}[v, u_0]$ (if $u = u_1$ and $v \neq v_1$) or $C' = P\overleftarrow{C}[v_0, u]uv_1\overrightarrow{C}[v_1, u_{-1}]u_{-1}u_1\overrightarrow{C}[u_1, u^-]u^-u_0$ (if $u \neq u_1$ and $v = v_1$) or $C' = P\overleftarrow{C}[v_0, u_1]u_1v_1\overrightarrow{C}[v_1, u_0]$ (if $u = u_1$ and $v = v_1$) is an o-cycle longer than C and contains all the vertices in C. By Lemma 1, there is a cycle containing all the vertices in $V(P) \cup V(C)$, a contradiction. $\square$

Claim 5. $d(u_0) + d(v_0) < n$.

Proof. Let $P' = u_0x_1x_2, \ldots, x_{r'}v_0$ be a (u_0, v_0)-path internally-disjoint with C such that its length is as large as possible.

Claim 5.1. $d_{G-C}(u_0) + d_{G-C}(v_0) \leq 2r'$.

Proof. We will show that all the neighbors of u_0 in $G-C$ are contained in $V(P')$. Assume not. Let x_1' be a neighbor of u_0, which is in $V(G-C)$ but not in $V(P')$. Obviously, we have $x_1u_1, x_1'u_1 \notin \widetilde{E}(G)$; otherwise there is a cycle longer than C by Lemma 1, contradicting to the choice of C. If $x_1x_1' \notin E(G)$, then $G[\{u_0, u_1, x_1, x_1'\}] \cong K_{1,3}$. Note that G is claw-o-heavy. Thus, we have $d(x_1)+d(x_1') \geq n$. It implies that either x_1 or x_1' is heavy. However, it follows from the fact G is claw-o-heavy and Lemma 2 that C is heavy, a contradiction. If $x_1x_1' \in E(G)$, then $P'' = u_0x_1'x_1P'[x_1, v_0]$ is a (u_0, v_0)-path internally-disjoint with C and longer than P', contradicting to the choice of P'. Therefore, it follows that $d_{G-C}(u_0) \leq r'$.

Similarly, we can obtain $d_{G-C}(v_0) \leq r'$ and the proof of this claim is complete. □

Let u_k be the last vertex on $\overrightarrow{C}[u_1, u_{r_1}]$ such that $u_0u_k \in E(G)$, u_l be the first vertex on $\overrightarrow{C}[u_{k+1}, u_{r_1}]$ such that $v_0u_l \in E(G)$.

Claim 5.2. For every vertex $u_{k'} \in N_{C[u_1,u_{k-1}]}(u_0) \cup \{u_0\}$, $v_0u_{k'+1} \notin E(G)$.

Proof. By Claim 3, we have $v_0u_1 \notin E(G)$. If $u_{k'} \neq u_0$, assume that $u_{k'}u_0 \in E(G)$ and $v_0u_{k'+1} \in E(G)$. Then $C' = Pv_0u_{k'+1}\overrightarrow{C}[u_{k'+1}, v_{-1}]v_{-1}v_1\overrightarrow{C}[v_1, u_{-1}]u_{-1}u_1\overrightarrow{C}[u_1, u_{k'}]u_{k'}u_0$ is an o-cycle containing all the vertices in $V(P) \cup V(C)$. Thus, there is a cycle longer than C by Lemma 1, a contradiction. □

Claim 5.3. $|[u_{k+1}, u_{l-1}]| = l-k-1 \geq r'$.

Proof. Assume that $|[u_{k+1}, u_{l-1}]| < r'$. Then $C' = Pv_0u_l\overrightarrow{C}[u_l, v_{-1}]v_{-1}v_1\overrightarrow{C}[v_1, u_{-1}]u_{-1}u_1$ $\overrightarrow{C}[u_1, u_k]u_ku_0$ is an o-cycle which contains all the vertices in $V(C) \setminus [u_{k+1}, u_{l-1}] \cup V(P')$ and $|V(C')| > c$. Hence there is a cycle longer than C by Lemma 1, a contradiction. □

The following claim is obvious.

Claim 5.4. $N_C(u_0) \cap [u_{k+1}, u_{r_1}] = \emptyset$.

Let $d_1 = |N_{C[u_1,u_k]}(u_0)|$. Then by Claim 5.4, $d_{C[u_1,u_{r_1}]}(u_0) = d_1$. By Claims 5.2 and 5.3, we have $d_{C[u_1,u_r]}(v_0) = d_{C[u_1,u_k]}(v_0) + d_{C[u_{k+1},u_{l-1}]}(v_0) + d_{C[u_l,u_{r_1}]}(v_0) \leq k - d_1 + r_1 - l + 1 \leq r_1 - d_1 - r'$. Thus $d_{C[u_1,u_{r_1}]}(u_0) + d_{C[u_1,u_{r_1}]}(v_0) \leq r_1 - r'$, and similarly, $d_{C[v_1,v_{r_2}]}(u_0) + d_{C[v_1,v_{r_2}]}(v_0) \leq r_2 - r'$. Hence $d_C(u_0) + d_C(v_0) \leq r_1 + r_2 - 2r' + 2 = c - 2r'$. Note that $d_{G-C}(u_0) + d_{G-C}(v_0) \leq 2r'$ by Claim 5.1. Therefore, $d(u_0) + d(v_0) \leq c < n$. □

Claim 6. Either $u_{-1}u_1 \in E(G)$ or $v_{-1}v_1 \in E(G)$.

Proof. Assume that $u_{-1}u_1 \notin E(G)$ and $v_{-1}v_1 \notin E(G)$. By Claim 2, we have $d(u_{-1}) + d(u_1) \geq n$ and $d(v_{-1}) + d(v_1) \geq n$. Thus, we obtain $d(u_{-1}) + d(v_{-1}) \geq n$ or $d(u_1) + d(v_1) \geq n$, contradicting to Lemma 3. □

We divide the left part of the proof into four cases.

Case 1. $S = P_6$.

If $u_0v_0 \notin E(G)$, then by Claim 4, $R_1 = G[\{u_0, u_{j_1-1}, u_{j_1}, v_0, v_{j_2-1}, v_{j_2}, w_1, w_2, \ldots, w_r\}] \cong P_{r+6}$. Since G is P_6-f-heavy, G is P_{6+r}-f-heavy. Note that C is heavy by Lemma 2. Hence each of $\{w_1, w_r\}$ is light. It follows from the fact $d_{R_1}(w_1, u_{j_1-1}) = d_{R_1}(w_r, v_{j_2-1}) = 2$ that each of $\{u_{j_1-1}, v_{j_2-1}\}$ is heavy. Then we have $u_{j_1-1}v_{j_2-1} \in \widetilde{E}(G)$, and it contradicts to Claim 4 (iii).

If $u_0v_0 \in E(G)$, then by Claim 4, $R_2 = G[\{u_0, u_{j_1-1}, u_{j_1}, v_0, v_{j_2-1}, v_{j_2}\}] \cong P_6$. By Claim 5, either u_0 or v_0 is light. Without loss of generality, assume that u_0 is light. Since G is P_6-f-heavy and $d_{R_2}(u_0, u_{j_1}) = d_{R_2}(u_0, v_{j_2-1}) = 2$, we have each of $\{u_{j_1}, v_{j_2-1}\}$ is heavy. It follows that $u_{j_1}v_{j_2-1} \in \widetilde{E}(G)$, which contradicts to Claim 4 (iii).

Case 2. $S = Z_2$

Case 2.1. $u_0v_0 \in E(G)$.

We claim that $r = 1$. If $r \geq 2$, then by the choice of P, we have $w_1v_0 \notin E(G)$ and $w_ru_0 \notin E(G)$. By Claims 1 and 3, we obtain $w_1u_1, w_rv_{-1}, u_1v_0, v_{-1}u_0 \notin \widetilde{E}(G)$. Hence $G[\{w_1, u_0, u_1, v_0\}] \cong K_{1,3}$ and $G[\{u_0, v_0, v_{-1}, w_r\}] \cong K_{1,3}$. Note that G is claw-o-heavy. It follows that $d(w_1) + d(v_0) \geq n$ and $d(w_r) + d(u_0) \geq n$. This means that either $d(w_1) + d(w_r) \geq n$ or $d(u_0) + d(v_0) \geq n$. However, we have $d(w_1) + d(w_r) < n$ since C is heavy. By Claim 5, we can obtain $d(u_0) + d(v_0) < n$, a contradiction.

By Claim 4, $R_1 = G[\{u_0, w_1, v_0, u_{j_1-1}, u_{j_1}\}] \cong Z_2$ and $R_2 = G[\{u_0, w_1, v_0, v_{j_2-1}, v_{j_2}\}] \cong Z_2$. Note that $d_{R_1}(w_1, u_{j_1-1}) = d_{R_2}(w_1, v_{j_2-1}) = 2$ and w_1 is light. It follows from the condition G is Z_2-f-heavy that each of $\{u_{j_1-1}, v_{j_2-1}\}$ is heavy. Thus, $u_{j_1-1}v_{j_2-1} \in \widetilde{E}(G)$, contradicting to Claim 4 (iii).

Case 2.2. $u_0v_0 \notin E(G)$.

By Claim 6, we have $u_{-1}u_1 \in E(G)$ or $v_{-1}v_1 \in E(G)$. Without loss of generality, suppose that $u_{-1}u_1 \in E(G)$. By Claims 1, 3 and the hypothesis that $u_0v_0 \notin E(G)$, we have $R_1 = G[\{w_1, w_2, u_{-1}, u_0, u_1\}] \cong Z_2$. Note that w_1 is light and $d_{R_1}(w_1, u_1) = d_{R_1}(w_1, u_{-1}) = 2$. It follows from the fact G is Z_2-f-heavy that each of $\{u_{-1}, u_1\}$ is heavy.

If $v_{-1}v_1 \notin E(G)$, then $G[\{w_r, v_{-1}, v_0, v_1\}] \cong K_{1,3}$ by Claim 1. Since G is claw-o-heavy, we have $d(v_{-1}) + d(v_1) \geq n$ by Claim 1. Hence we can obtain $d(u_{-1}) + d(u_1) \geq n$ or $d(v_{-1}) + d(v_1) \geq n$, which contradicts to Lemma 3.

If $v_{-1}v_1 \in E(G)$, then $R_1 = G[\{w_{r-1}, w_r, v_{-1}, v_0, v_1\}] \cong Z_2$. Note that w_r is light and $d_{R_1}(w_r, v_{-1}) = d_{R_1}(w_r, v_1) = 2$. It follows from the fact G is Z_2-f-heavy that $d(v_{-1}) \geq n/2$ and $d(v_1) \geq n/2$. Hence $d(u_{-1}) + d(v_{-1}) \geq n$, contradicting to Lemma 3.

Case 3. $S = W$ or $S = N$.

When $S = W$ and $u_0v_0 \in E(G)$, similarly as Case 2.1, we can prove that $r = 1$. By Claim 4, $R_1 = G[\{u_0, w_1, v_0, u_{j_1-1}, u_{j_1}, v_{j_2-1}\}] \cong W$. Note that $d_{R_1}(w_1, u_{j_1-1}) = d_{R_1}(w_1, v_{j_2-1}) = 2$ and w_1 is light. Since G is W-f-heavy, it follows that each of $\{u_{j_1-1}, v_{j_2-1}\}$ is heavy. Thus, $u_{j_1-1}v_{j_2-1} \in \widetilde{E}(G)$, contradicting to Claim 4 (iii).

Now we can suppose that $S = W$ and $u_0v_0 \notin E(G)$ or $S = N$.

By Claim 6, we have $u_{-1}u_1 \in E(G)$ or $v_{-1}v_1 \in E(G)$. Suppose, without loss of generality, that $u_{-1}u_1 \in E(G)$. Note that $G[u_{-1}, u_1]$ is (u_{-1}, u_0, u_1)-composed.

Claim 7. If $G[u_{-k}, u_l]$ is (u_{-k}, u_0, u_l)-composed with the canonical ordering $u_{-k}, u_{-k+1}, \dots, u_{l-1}, u_l$, then $k \leq r_2 - 2$ and $l \leq r_1 - 2$.

Proof. Suppose that $k \geq r_2 - 1$. Let D_1, D_2,..., D_r be a canonical sequence of $G[u_{-k}, u_l]$ corresponding to the canonical ordering $u_{-k}, u_{-k+1}, \dots, u_{l-1}, u_l$. Consider the graph $D' = D_{\widehat{-r_2+1}}$, where $\widehat{-r_2+1}$ be the smallest integer such that $u_{-r_2+1} \in D_{\widehat{-r_2+1}}$. Note that the index $\widehat{-r_2+1}$ exists since $0 \geq -r_2+1 \geq -k$. By Lemma 4, there is a (u_0, u_l)-path P' satisfying $V(P') = [u_{-r_2+1}, u_l]$. Then $C' = v_1v_0P[v_0, u_0]P'[u_0, u_l]\overrightarrow{C}[u_l, v_{-1}]v_{-1}v_1$ is an o-cycle such that $V(C) \cup V(P) \subseteq V(C')$, and there is a cycle longer than C by Lemma 1, a contradiction. Similarly, we can prove that $l \leq r_1 - 2$. $\square$

Claim 8. If $G[u_{-k}, u_l]$ is (u_{-k}, u_0, u_l)-composed with the canonical ordering $u_{-k}, u_{-k+1}, \dots, u_{l-1}, u_l$, where $k \leq r_2 - 2$ and $l \leq r_1 - 2$, and moreover the following two statements hold:
(i) there is not a heavy pair in $G[u_{-k-1}, u_{l+1}]$,
(ii) there is not a heavy triangle in $G[u_{-k-1}, u_{l+1}]$,
then one of the following is true:
(1) $G[u_{-k-1}, u_l]$ is (u_{-k-1}, u_0, u_l)-composed with the canonical ordering $u_{-k-1}, u_k, \dots, u_l$,
(2) $G[u_{-k}, u_{l+1}]$ is (u_{-k}, u_0, u_{l+1})-composed with the canonical ordering $u_{-k}, u_k, \dots, u_{l+1}$,
(3) $G[u_{-k-1}, u_{l+1}]$ is (u_{-k-1}, u_0, u_{l+1})-composed with the canonical ordering $u_{-k-1}, u_k, \dots, u_{l+1}$.

Proof. Assume not. Then we have $u_{-k-1}u_s \notin E(G)$ for every vertex $s \in [-k+1, l]$, $u_su_{l+1} \notin E(G)$ for every vertex $s \in [-k, l-1]$, and $u_{-k-1}u_{l+1} \notin E(G)$.

Claim 8.1. For any vertex $u_s \in [u_{-k-1}, u_{l+1}] \setminus \{u_0\}$ and $w \in \{w_1, w_2\}$, we have $u_sw \notin \widetilde{E}(G)$. Moreover, we have $u_0w_2 \notin E(G)$ if $u_0v_0 \notin E(G)$.

Proof. Without loss of generality, suppose that $u_sw \in \widetilde{E}(G)$ and $s > 0$. If $s = 1$, then $u_1w \notin \widetilde{E}(G)$ by Claim 1 or 3. Now assume that $s \in [2, l+1]$. Since $G[u_{-k}, u_l]$ is (u_{-k}, u_0, u_l)-composed, there exists an integer $t \in [-k, -1]$ such that $G[u_t, u_{s-1}]$ is

(u_t, u_0, u_{s-1})-composed. Hence there is a (u_0, u_t)-path P' such that $V(P') = [u_t, u_{s-1}]$. Let $C' = P'[u_0, u_t]\overleftarrow{C}[u_t, u_s]u_s wP[w, u_0]$ (if $w \neq v_0$) or $C' = P'[u_0, u_t]\overleftarrow{C}[u_t, v_1]v_1 v_{-1}\overleftarrow{C}[v_{-1}, u_s]u_s v_0$ $P[v_0, u_0]$ (if $w = v_0$). Clearly, C' is an o-cycle such that $V(C) \subseteq V(C')$ and $|V(C')| > |V(C)|$. By Lemma 1, a contradiction.

Moreover if $r \geq 2$, then $u_0 w_2 \notin E(G)$ by the choice of P. If $r = 1$ and $u_0 v_0 \notin E(G)$, then $u_0 w_2 = u_0 v_0 \notin E(G)$. $\square$

Let $G' = G[[u_{-k-1}, u_l] \cup \{w_1, w_2\}]$ and $G'' = G[[u_{-k-1}, u_{l+1}] \cup \{w_1, w_2\}]$.

Claim 8.2. If $S = W$ and $u_0 v_0 \notin E(G)$, then G'' and G' are $\{K_{1,3}, W\}$-free; If $S = N$, then G'' and G' are $\{K_{1,3}, N\}$-free.

Proof. Note that G' is an induced subgraph of G''. Hence we only need to prove that G'' satisfies the required property.

Assume that G'' contains an induced claw. Without loss of generality, let H be the claw. If $V(H) \subseteq [u_{-k-1}, u_{l+1}]$, then since G is claw-o-heavy, there is a heavy pair in $[u_{-k-1}, u_{l+1}]$, which contradicts to condition (i) of Claim 8. If $w_1 \in V(H)$ or $w_2 \in V(H)$, then by Claim 8.1, $d_H(w_1) \leq d_{G''}(w_1) = 2$ and $d_H(w_2) \leq d_{G''}(w_2) \leq 2$. Hence u_0 is the center of H and the other two end vertices x_1, x_2 of H are in $[u_{-k-1}, u_{l+1}]$. By Claim 8.1, $w_1 x_1, w_1 x_2 \notin \widetilde{E}(G)$. Since G is claw-o-heavy, x_1, x_2 is heavy pair in $G[u_{-k-1}, u_{l+1}]$, a contradiction.

If $S = W$, then assume that G'' contains an induced subgraph $H \cong W$ depicted in Figure 1. Obviously, one vertex of $\{a_1, c_1\}$ and one vertex of $\{a_2, b_1\}$ and one vertex of $\{a_3, b_2\}$ are heavy. Hence there are at least three heavy vertices in G''. By Lemma 2 and the choice of C, w_1 is not heavy. Thus there is at least one heavy vertex in $[u_{-k-1}, u_{l+1}] \backslash \{u_0\}$. By Claim 8.1, w_2 is not heavy. Thus there are at least three heavy vertices in $[u_{-k-1}, u_{l+1}]$. If these three heavy vertices are adjacent to each other, then there is a heavy triangle in $[u_{-k-1}, u_{l+1}]$, a contradiction. Otherwise, there is a heavy pair in $[u_{-k-1}, u_{l+1}]$, a contradiction.

If $S = N$, then assume that G'' contains an induced subgraph $H \cong N$ depicted in Figure 1. Obviously, one vertex of $\{a_1, b_2\}$ and one vertex of $\{a_2, b_3\}$ and one vertex of $\{a_3, b_1\}$ are heavy. Hence there are at least three heavy vertices in G''. Similarly as the analysis above, we can deduce a contradiction.

The proof is complete. $\square$

Now, we define $N_i = \{x \in V(G') : d_{G'}(x, u_{-k-1}) = i\}$. Therefore, we have $N_0 = \{u_{-k-1}\}$, and $N_1 = \{u_{-k}\}$ by the fact that $u_{-k-1} u_s \notin E(G)$, where $s \in [-k+1, l]$.

Without loss of generality, we assume $u_0 \in N_j$, where $j \geq 2$. Then we have $w_1 \in N_{j+1}$, $w_2 \in N_{j+1}$ if $u_0 w_2 \in E(G)$, and $w_2 \in N_{j+2}$ if $u_0 w_2 \notin E(G)$ by Claim 8.1.

Claim 8.3. For $i \in [1, j]$, N_i is a clique.

Proof. Suppose that $|N_i| = 1$ for some $i \in [2, j-1]$, and we set $N_i = \{x\}$. Then x is a cut vertex of $G[u_{-k}, u_l]$, contradicting to the fact that $G[u_{-k}, u_l]$ is 2-connected. Thus, we have $|N_i| \geq 2$ for every integer $i \in [2, j-1]$.

Now, we prove this claim by induction on i. If $i = 1$, it is trivially true. If $i = 2$, suppose that there exist $x, y \in N_2$ such that $xy \notin E(G)$, then $G[\{x, y, u_{-k}, u_{-k-1}\}] \cong K_{1,3}$, a contradiction. Hence the claim is true when $i = 2$. Now, we assume $3 \leq i \leq j$, and we have each of N_{i-3}, N_{i-2}, N_{i-1}, N_{i+1} is nonempty, and $|N_{i-1}| \geq 2$.

Case A. $i < j$ or $i = j$ and $w_2 u_0 \notin E(G)$.

Note that N_{i+2} is nonempty in this case. Let x be a vertex of N_i such that y is a neighbor of it in N_{i+1} which has a neighbor z in N_{i+2}. For every vertex $x' \in N_i \backslash \{x\}$, we will show that $xx' \in E(G)$. Assume that $xx' \notin E(G)$. If $x'y \in E(G)$, then $G[\{x, x', y, z\}] \cong K_{1,3}$, a contradiction. If x and x' have a common neighbor in N_{i-1}, let it be v and w be a neighbor of v in N_{i-2}. Then $G[\{x, x', v, w\}] \cong K_{1,3}$, a contradiction. Thus we assume x and x' have no common neighbors in N_{i-1}.

Let v be a neighbor of x in N_{i-1} and v' be a neighbor of x' in N_{i-1}. By induction hypothesis, we have $vv' \in E(G)$. Let w be a neighbor of v in N_{i-2} and u be a neighbor of w in N_{i-3}. If $wv' \notin E(G)$, then $G[\{x, v, v', w\}] \cong K_{1,3}$, a contradiction. Hence it follows that $v'w \in E(G)$. Now, we have $G[\{y, x, x', v, v', w\}] \cong W$ and $G[\{x, x', v, v', w, u\}] \cong N$, a contradiction to Claim 8.2. Therefore, for every vertex $x' \in N_i \setminus \{x\}$, $xx' \in E(G)$.

If there exist $x', x'' \in N_i \setminus \{x\}$ such that $x' \neq x''$ and $x'x'' \notin E(G)$, then we have $xx' \in E(G)$ and $xx'' \in E(G)$. Note that y is a neighbor of x in N_{i+1}. If $x'y \in E(G)$ or $x''y \in E(G)$, then similar to the case of x given above, we have $x'x'' \in E(G)$, a contradiction. It follows that $x'y \notin E(G)$ and $x''y \notin E(G)$. Therefore, $G[\{x, x', x'', y\}] \cong K_{1,3}$, a contradiction.

Case B. $i = j$ and $w_2 u_0 \in E(G)$.

We prove that for every vertex $x \in N_j \setminus \{u_0\}$, $xu_0 \in E(G)$. Assume that $xu_0 \notin E(G)$. If x and u_0 have a common neighbor in N_{j-1}, let it be v and w be a neighbor of v in N_{j-2}. Then $G[\{x, u_0, v, w\}] \cong K_{1,3}$, a contradiction. Thus we assume x and u_0 have no common neighbors in N_{j-1}.

Let v' be a neighbor of u_0 in N_{j-1} and v be a neighbor of x in N_{j-1}. By induction hypothesis, we have $vv' \in E(G)$. Note that $u_0 v \notin E(G)$ and $xv' \notin E(G)$. Let w' be a neighbor of v' in N_{i-2} and u' be a neighbor of w' in N_{j-3}. If $w'v \notin E(G)$, then

$G[\{u_0, v, v', w'\}] \cong K_{1,3}$, a contradiction. Hence it follows that $vw' \in E(G)$. Now we have $G[\{u_0, x, v, v', w', u'\}] \cong N$ and $G[\{u_0, x, v, v', w', w_1\}] \cong W$, contradicting to Claim 8.2. Therefore, for every vertex $x \in N_j \setminus \{u_0\}$, $xu_0 \in E(G)$.

If there exist $x', x'' \in N_j \setminus \{u_0\}$ such that $x' \neq x''$ and $x'x'' \notin E(G)$. By the analysis above, we have $u_0x' \in E(G)$ and $u_0x'' \in E(G)$. Note that $x', x'' \neq w_2$. By Claim 8.1, we have $w_1x' \notin E(G)$ and $w_1x'' \notin E(G)$. Hence $G[\{w_1, u_0, x', x''\}] \cong K_{1,3}$, a contradiction.

The proof is complete. □

Claim 8.4. If $S = W$ and $u_0v_0 \notin E(G)$, then $N_{G'}(u_0) \setminus \{w_1\}$ is a clique; If $S = N$, then $N_{G'}(u_0) \setminus \{w_1, w_2\}$ is a clique.

Proof. Suppose not. If $S = W$ and $u_0v_0 \notin E(G)$, then let $x, y \in N_{G'}(u_0) \setminus \{w_1\}$ are two vertices such that $xy \notin E(G)$. By Claim 8.1, we have $x, y \neq w_2$, $w_1x \notin E(G)$ and $w_1y \notin E(G)$. Hence we have $G[\{x, y, u_0, w_1\}] \cong K_{1,3}$, a contradiction. If $S = N$, then suppose $x, y \in N_{G'}(u_0) \setminus \{w_1, w_2\}$ are two vertices such that $xy \notin E(G)$. Hence we have $G[\{x, y, u_0, w_1\}] \cong K_{1,3}$ by Claim 8.1, a contradiction. □

Claim 8.5. $[u_{-k}, u_l] \subseteq \bigcup_{i=1}^{j} N_i$.

Proof. Assume there exists a vertex $x \in [u_{-k}, u_l]$ such that $x \in N_{j+1}$. Let y be a neighbor of x in N_j, z be a neighbor of u_0 in N_{j-1} and v be a neighbor of z in N_{j-2}. Note that $x, z \notin \{w_1, w_2\}$. Then we have $xu_0 \notin E(G)$, since otherwise $xz \in E(G)$ by Claim 8.4, and it implies that $x \notin N_{j+1}$, a contradiction. By Claim 8.1, we have $yw_1 \notin E(G)$. Note that $yu_0 \in E(G)$ by Claim 8.3. If $yz \notin E(G)$, then $G[\{y, z, u_0, w_1\}] \cong K_{1,3}$, a contradiction. Now we assume $yz \in E(G)$. If $S = W$ and $u_0v_0 \notin E(G)$, we have $u_0w_2 \notin E(G)$ by Claim 8.1. Hence $G[\{x, y, z, u_0, w_1, w_2\}] \cong W$, and it contradicts to Claim 8.2. If $S = N$, then $G[\{x, y, z, u_0, w_1, v\}] \cong N$, which also contradicts to Claim 8.2.

The proof is complete. □

It follows from Claim 8.5 that $u_l \in N_j$ or $u_l \in N_i$ where $i \in [2, j-1]$.

If $u_l \in N_j$, then let x be a neighbor of u_0 in N_{j-1} and y be a neighbor of x in N_{j-2}. Since $u_l, u_0 \in N_j$, we have $u_lu_0 \in E(G)$ by Claim 8.3. By Claim 8.1, we have $u_lw_1 \notin E(G)$ and $xw_1 \notin E(G)$. If $u_lx \notin E(G)$, then $G[\{w_1, u_0, u_l, x\}] \cong K_{1,3}$, a contradiction. Otherwise, we have $u_lx \in E(G)$. If $S = W$ and $u_0v_0 \notin E(G)$, we have $u_0w_2 \notin E(G)$ by Claim 8.1. By Claim 8.1, $G[\{x, y, u_0, u_l, w_1, w_2\}] \cong W$. If $S = N$, then by the fact that $u_su_{l+1} \notin E(G)$, where $s \in [-k, l-1]$ and Claim 8.1, we have $G[\{x, y, u_0, u_l, u_{l+1}, w_1\}] \cong N$. In each case, it contradicts to Claim 8.2.

Now assume that $u_l \in N_i$, where $i \in [2, j-1]$ and $j \geq 3$. If u_l has a neighbor in N_{i+1}, without loss of generality, let x be a required vertex and y be a neighbor of u_l in N_{i-1}. Note that $i+1 \leq j$, and it implies that $x \neq w_1, w_2$. By the fact that $u_s u_{l+1} \notin E(G)$, where $s \in [-k, l-1]$, we have $G[\{u_l, u_{l+1}, x, y\}] \cong K_{1,3}$, a contradiction. Then we assume u_l has no neighbors in N_{i+1}.

Since $|N_i| \geq 2$, we can choose $x \in N_i$ be a vertex other than u_l such that y is a neighbor of x in N_{i+1} which has a neighbor z in N_{i+2}. Let u be a neighbor of x in N_{i-1} and v be a neighbor of u in N_{i-2}. Note that $u_l x \in E(G)$ by Claim 8.3. If $u_l u \notin E(G)$, then $G[\{x, y, u_l, u\}] \cong K_{1,3}$, a contradiction. Thus we have $u_l u \in E(G)$. Hence we have $G[\{x, y, z, u_l, u, v\}] \cong W$ and $G[\{x, y, u_l, u_{l+1}, u, v\}] \cong N$, contradicting to Claim 8.2.

The proof is complete. $\square$

Now we choose k and l such that

(1) $G[u_{-k}, u_l]$ is (u_{-k}, u_0, u_l)-composed with the canonical ordering $u_{-k}, u_{-k+1}, \ldots, u_l$;

(2) there is not a heavy pair in $G[u_{-k}, u_l]$;

(3) there is not a heavy triangle in $G[u_{-k}, u_l]$; and

(4) $k+l$ is as large as possible.

By Claim 8, we know one of the following cases occurs:

(a) there exists a vertex $u_{s'} \in [u_{-k+1}, u_l]$ such that $u_{-k-1}u_{s'} \notin E(G)$ and $d(u_{-k-1}) + d(u_{s'}) \geq n$.

(b) there exists a vertex $u_s \in [u_{-k+1}, u_l]$ such that $u_{-k-1}u_s \in E(G)$ and each of $\{u_s, u_{-k-1}\}$ is heavy.

(c) there exists a vertex $u_{t'} \in [u_{-k}, u_{l-1}]$ such that $u_{l+1}u_{t'} \notin E(G)$ and $d(u_{l+1}) + d(u_{t'}) \geq n$.

(d) there exists a vertex $u_t \in [u_{-k}, u_{l-1}]$ such that $u_{l+1}u_t \in E(G)$ and each of $\{u_{l+1}, u_t\}$ is heavy.

(e) $u_{-k-1}u_{l+1} \notin E(G)$ and $d(u_{-k-1}) + d(u_{l+1}) \geq n$.

(f) $u_{-k-1}u_{l+1} \in E(G)$ and each of $\{u_{-k-1}, u_{l+1}\}$ is heavy.

Hence there exists a vertex $u_i \in [u_{-k+1}, u_l]$ such that $d(u_{-k-1}) + d(u_i) \geq n$, or there exists a vertex $u_j \in [u_{-k}, u_{l-1}]$ such that $d(u_{l+1}) + d(u_j) \geq n$, or $d(u_{-k-1}) + d(u_{l+1}) \geq n$.

Claim 9. (u_{-k-1}, u_l) or (u_{-k}, u_{l+1}) or (u_{-k-1}, u_{l+1}) is u_0-good on C.

Proof. If there exists a vertex $u_i \in [u_{-k+1}, u_l]$ such that $d(u_{-k-1}) + d(u_i) \geq n$, then since $G[u_{-k}, u_l]$ is (u_{-k}, u_0, u_l)-composed with the canonical ordering $u_{-k}, u_{-k+1}, \ldots, u_l$, there exists a (u_0, u_l)-path P such that $V(P) = [u_{-k-1}, u_l] \setminus \{u_{-k-1}\}$. Moreover, there is a (u_0u_l, u_iu_{-k})-pair D such that $V(D) = [u_{-k}, u_l]$, then $D' = D + u_{-k}u_{-k-1}$ is a (u_0u_l, u_iu_{-k-1})-pair D' such that $V(D') = [u_{-k-1}, u_l]$. Therefore (u_{-k-1}, u_l) is u_0-good on C.

If there exists a vertex $u_j \in [u_{-k}, u_{l-1}]$ such that $d(u_{l+1}) + d(u_j) \geq n$, then we can prove this claim similarly.

Now suppose $d(u_{-k-1})+d(u_{l+1}) \geq n$. By Lemma 4, there exists a (u_0, u_l)-path P' such that $V(P') = [u_{-k}, u_l]$ and a (u_0, u_{-k})-path P'' such that $V(P'') = [u_{-k}, u_l]$. Then $P = P'u_l u_{l+1}$ is a (u_0, u_{l+1})-path such that $V(P) = [u_{-k}, u_{l+1}]$, and $D = P''u_{-k}u_{-k-1} \cup u_{l+1}$ is a $(u_0 u_{l+1}, u_{l+1}u_{-k-1})$-pair such that $V(D) = [u_{-k-1}, u_{l+1}]$. Thus (u_{-k-1}, u_{l+1}) is u_0-good on C.

The proof is complete. $\square$

Claim 10. There exists a vertex $v_{-k'} \in V(\overrightarrow{C}[v_1, u_{-k-1}])$ and $v_{l'} \in V(\overleftarrow{C}[v_{-1}, u_{l+1}])$ with $(v_{-k'}, v_{l'})$ is v_0-good.

Proof. If $v_{-1}v_1 \notin E(G)$, then $d(v_{-1}) + d(v_1) \geq n$ by Claim 2. Moreover, $P = v_0 v_{-1}$ is a $(v_0 v_{-1})$-path and $D = v_0 v_1 \cup \{v_{-1}\}$ is a $(v_0 v_{-1}, v_{-1}v_1)$-pair. Hence (v_{-1}, v_1) is v_0-good.

If $v_{-1}v_1 \in E(G)$, then $G[v_{-1}, v_1]$ is (v_{-1}, v_0, v_1)-composed.

Now, set $r_1' = r_1 - l$ and $r_2' = r_2 - k$, where $k \leq r_2 - 2$ and $l \leq r_1 - 2$ by Claim 7. Similar to Claims 7 and 8, we have Claims 10.1 and 10.2 as follows.

Claim 10.1. If $G[v_{-k'}, v_{l'}]$ is $(v_{-k'}, v_0, v_{l'})$-composed with the canonical ordering $v_{-k'}, v_{-k'+1}, \ldots, v_{l'}$, then $k' \leq r_1' - 1$ and $l' \leq r_2' - 1$.

Claim 10.2. If $G[v_{-k'}, v_{l'}]$ is $(v_{-k'}, v_0, v_{l'})$-composed with the canonical ordering $v_{-k'}, v_{-k'+1}, \ldots, v_{l'}$, where $k' \leq r_1' - 1$ and $l' \leq r_2' - 1$, and moreover the following two statements hold:
(i) there is not a heavy pair in $G[v_{-k'-1}, v_{l'+1}]$,
(ii) there is not a heavy triangle in $G[v_{-k'-1}, v_{l'+1}]$,
then one of the following is true:
(1) $G[v_{-k'-1}, v_{l'}]$ is $(v_{-k'-1}, v_0, v_{l'})$-composed with the canonical ordering $v_{-k'-1}, v_{-k'}, \ldots, v_{l'}$,
(2) $G[v_{-k'}, v_{l'+1}]$ is $(v_{-k'}, v_0, v_{l'+1})$-composed with the canonical ordering $v_{-k'}, v_{-k'+1}, \ldots, v_{l'+1}$,
(3) $G[v_{-k'-1}, v_{l'+1}]$ is $(v_{-k'-1}, v_0, v_{l'+1})$-composed with the canonical ordering $v_{-k'-1}, v_{-k'}, \ldots, v_{l'+1}$.

Now we choose k' and l' such that
(1) $G[v_{-k'}, v_{l'}]$ is $(v_{-k'}, v_0, v_{l'})$-composed with the canonical ordering $v_{-k'}, v_{-k'+1}, \ldots, v_{l'}$;
(2) there is not a heavy pair in $G[v_{-k'}, v_{l'}]$;
(3) there is not a heavy triangle in $G[v_{-k'}, v_{l'}]$; and
(4) $k' + l'$ is as large as possible.

Similar to Claim 9, $(v_{-k'-1}, v_{l'})$ or $(v_{-k'}, v_{l'+1})$ or $(v_{-k'-1}, v_{l'+1})$ is v_0-good on C. $\square$

By Claims 9 and 10, there exists a cycle which contains all the vertices in $V(C) \cup V(P)$ by Lemma 5, a contradiction.

The proof is complete. $\square$

4 Concluding Remarks

It is known that Faudree and Gould [9] extended Bedrossian's result to 2-connected graphs on $n \geq 10$ vertices.

Theorem 15 (Faudree and Gould [9])**.** *Let R and S be connected graphs other than P_3 and let G be a 2-connected graph on $n \geq 10$ vertices. Then G being $\{R, S\}$-free implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, Z_3, B$, N or W.*

Chen et al. [6] showed every 2-connected $\{K_{1,3}, Z_3\}$-f-heavy graph on $n \geq 10$ vertices is Hamiltonian.

Theorem 16 (Chen, Wei and Zhang [6])**.** *Let G be a 2-connected graph on $n \geq 10$ vertices. If G is $\{K_{1,3}, Z_3\}$-f-heavy, then G is Hamiltonian.*

Together with Theorems 12, 16 and Remark 1, we have the following result which extends Theorem 15.

Theorem 17. *Let R and S be connected graphs other than P_3 and let G be a 2-connected graph on $n \geq 10$ vertices. Then G being $\{R, S\}$-f-heavy implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$ or W.*

Li et al. [10] also constructed a class of 2-connected graphs on $n \geq 10$ vertices which are $\{K_{1,3}, Z_3\}$-o-heavy but not Hamiltonian. Thus it is natural to pose the following problem.

Problem 4. Is every 2-connected claw-o-heavy and Z_3-f-heavy graph on $n \geq 10$ vertices Hamiltonian?

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